Thermal Instability in Plasma with Finite Larmor Radius

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The effects of the finite Larmor radius of the ions on the thermal instability of a plasma are investigated. When the instability sets in as stationary convection, the finite Larmor radius is found to have a stabilizing effect. The conditions for the nonexistence of overstability are investigated. The case with horizontal magnetic field is discussed.

1. Introduction

The thermal instability of a fluid layer heated from below has been discussed by Chandrasekhar[1] under varying assumptions of hydromagnetics. The effects of the finiteness of the ion Larmor radius, which exhibit itself in the form of a magnetic viscosity in the fluid equations, have been studied by many authors[2-5]. Melchior and Popowich[6] have considered the finite Larmor radius effect on the Kelvin-Helmholtz instability of a fully ionized plasma while that on the Rayleigh-Taylor instability has been studied by Singh and Hans[7]. Recently, the author[8] has studied the finite Larmor radius and Hall effects on the thermal instability of a rotating plasma. The effect of the finite Larmor radius on the thermal instability of a plasma in the presence of a vertical magnetic field has also been studied by the author[9].

It seems of some interest to study the effect of the finite Larmor radius on the thermal instability of a plasma in the form of an infinite, horizontal layer of thickness d, acted on by a horizontal magnetic field $\bar{H}(H, 0, 0)$ and gravity force $\bar{g}(0, 0, -g)$. This layer is heated from below such that a steady adverse temperature gradient $\beta (= |\mathrm{d}T/\mathrm{d}z|)$ is maintained. The plasma is assumed to be incompressible and of finite electrical conductivity.

2. Perturbation Equations and the Characteristic Value Problem

The linearized hydromagnetic perturbation equations appropriate to the problem are:

$$\varrho \frac{\partial \bar{q}}{\partial t} = -\nabla \delta \mathbf{P} + \varrho \, \nu \, \nabla^2 \bar{q}$$

$$+ \frac{\mu_e}{4\pi} (\nabla \times \bar{h}) \times \bar{H} + \bar{g}\, \delta\varrho, \qquad (1)$$

$$\nabla \cdot \bar{q} = 0, \qquad (2)$$

$$\frac{\partial \bar{h}}{\partial t} = (\bar{H} \cdot \nabla) \, \bar{q} + \eta \, \nabla^2 \, \bar{h}, \quad \nabla \cdot \bar{h} = 0, \qquad (3)$$

$$\frac{\partial \Theta}{\partial t} + (\bar{q} \cdot \nabla) \, T = \varkappa \, \nabla^2 \, \Theta, \qquad (4)$$

where $\bar{q}(u, v, w)$, $\bar{h}$, $\delta\mathbf{P}$, Θ, $\delta\varrho$ denote, respectively, the perturbations in velocity, magnetic field, pressure $\mathbf{P}$, temperature T and density ϱ. g, $\varkappa$, ν, μ_e and η denote, respectively, the gravitational acceleration, the thermal diffusivity, the kinematic viscosity, the magnetic permeability and the resistivity.

For the magnetic field along the x-axis the stress tensor $\mathbf{P}$, taking into account the finite ion gyration radius[3], has the components

$$P_{xx} = p, \quad P_{xy} = P_{yx} = -2\varrho\,\nu_0 \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right),$$

$$P_{xz} = P_{zx} = 2\varrho\,\nu_0 \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right),$$

$$P_{yy} = p - \varrho\,\nu_0 \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), \qquad (5)$$

$$P_{yz} = P_{zy} = \varrho\,\nu_0 \left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right),$$

$$P_{zz} = p + \varrho\,\nu_0 \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right),$$

where p is the scalar part of pressure and $\varrho\,\nu_0 = N\,T/4\,\omega_\mathrm{H}$, where ω_H is ion gyration frequency, while N and T denote, respectively, the number density and temperature of the ions.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y, and t is given by

$$\exp\{i\,k_x\,x + i\,k_y\,y + n\,t\}, \qquad (6)$$

where k_x, k_y are wave numbers along the x and y directions respectively and n is the frequency.

Using (5) and (6) and taking $\delta \varrho = -\alpha \varrho \Theta$ in   Boussinesq approximation we get, on eliminating δp:

$$n\left(\frac{d^2}{dz^2} - k^2\right) w - \nu \left(\frac{d^2}{dz^2} - k^2\right)^2 w + g k^2 \alpha \Theta - \frac{\mu_e H}{4 \pi \varrho} i k_x \left(\frac{d^2}{dz^2} - k^2\right) h_z \tag{7}$$

$$+ \nu_0 \left[i k_y \left(\frac{d^2}{dz^2} - 4 k_x^2 - k_y^2\right) \frac{dw}{dz} - 2 i k_x \frac{d^2 \zeta}{dz^2} - k^2 \left(\frac{d^2}{dz^2} - 2 k_x^2 - k_y^2\right) v + 2 k_x k_y k^2 u \right] = 0 ,$$

where $\zeta = i k_x v - i k_y u$ denotes the z-component of the vorticity. α is the coefficient of thermal expansion.

Taking the curl of (1), we get

$$\left[n - \nu \left(\frac{d^2}{dz^2} - k^2\right)\right] \zeta = - i k_x \nu_0 \left(\frac{d^2}{dz^2} + 2 k_x^2 - k_y^2\right) w + \frac{\mu_e H}{4 \pi \varrho} i k_x \xi , \tag{8}$$

where $\xi = i k_x h_y - i k_y h_x$ denotes the z-component of the vector curl $\bar{h}$.

Similarly, taking the curl of (3), we have

$$n \xi = (i k_x H) \zeta + \eta \nabla^2 \xi . \tag{9}$$

Equations (4) and (3) can be written as

$$[n - \varkappa(d^2/dz^2 - k^2)] \Theta = \beta w , \tag{10}$$

$$[n - \eta (d^2/dz^2 - k^2)] h_z = i k_x H w , \tag{11}$$

Letting $a = k d$, $\sigma = n d^2/\nu$, $p_1 = \nu/\varkappa$ and $p_2 = \nu/\eta$,   Equations (7) − (11) become

$$(D^2 - a^2)(D^2 - a^2 - \sigma) w - \left(\frac{g \alpha d^2}{\nu}\right) a^2 \Theta + i k_x \frac{\mu_e H d^2}{4 \pi \varrho \nu} (D^2 - a^2) h_z +$$

$$+ \left(\frac{i k_x \nu_0 d^2}{\nu}\right) \left(D^2 - a^2 + 3 \frac{k_x^2}{k^2} a^2\right) \zeta = 0 , \tag{12}$$

$$(D^2 - a^2 - \sigma) \zeta = \left(\frac{i k_x \nu_0}{\nu}\right) \left(D^2 - a^2 + 3 \frac{k_x^2}{k^2} a^2\right) w - i \frac{\mu_e H k_x d^2}{4 \pi \varrho \nu} \xi , \tag{13}$$

$$(D^2 - a^2 - p_2 \sigma) \xi = - \left(\frac{i k_x H d^2}{\eta}\right) \zeta , \tag{14}$$

$$(D^2 - a^2 - p_1 \sigma) \quad = - \left(\frac{\beta d^2}{\varkappa}\right) w \quad \text{and} \quad (D^2 - a^2 - p_2 \sigma) h_z = - \left(\frac{i k_x H d^2}{\eta}\right) w , \quad (15, 16)$$

where $$D = d \frac{\mathrm{d}}{\mathrm{d}z} .$$

3. The Case of Stationary Convection

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. In this case Eqs. (12) − (16) reduce to

$$(D^2 - a^2)^2 w - \left(\frac{g \alpha d^2}{\nu}\right) a^2 \Theta + \frac{i k_x \mu_e H d^2}{4 \pi \varrho \nu} (D^2 - a^2) h_z + \left(\frac{i k_x \nu_0 d^2}{\nu}\right) \left(D^2 - a^2 + 3 \frac{k_x^2}{k^2} a^2\right) \zeta = 0 , \tag{17}$$

$$(D^2 - a^2) \zeta = \left(\frac{i k_x \nu_0}{\nu}\right) \left(D^2 - a^2 + 3 \frac{k_x^2}{k^2} a^2\right) w - i k_x \frac{\mu_e H d^2}{4 \pi \varrho \nu} \xi , \tag{18}$$

$$(D^2 - a^2) \xi = - \left(\frac{i k_x H d^2}{\eta}\right) \zeta , \tag{19}$$

$$(D^2 - a^2)\,\Theta = -\left(\frac{\beta\,d^2}{\varkappa}\right) w\,, \qquad (D^2 - a^2)\,h_z = -\left(\frac{i\,k_x H\,d^2}{\eta}\right) w\,. \qquad (20,\,21)$$

Eliminating Θ, h_z, ξ, ζ from Eqs. $(17) - (21)$, we get

$$[\,(D^2 - a^2)^2 + a^2\,Q\cos^2\Phi\,]\,[\,(D^2 - a^2)^3 + R\,a^2 + a^2\,Q\cos^2\Phi\,(D^2 - a^2)\,]\,w$$
$$= \left(\frac{a^2\,\nu_0{}^2\cos^2\Phi}{\nu^2}\right)(D^2 - a^2)^2\,(D^2 - a^2\,+ 3\,a^2\cos^2\Phi)^2\,w\,, \qquad (22)$$

where $R = g\,\alpha\,\beta\,d^4/\nu\,\varkappa$ is the Rayleigh number, $Q = \mu_e\,H^2\,d^2/4\,\pi\,\varrho\,\nu\,\eta$ is Chandrasekhar number and $\cos\Phi = k_x/k$.

This tenth degree equation with suitable boundary conditions constitutes the required characteristic value problem.

We consider the case of a fluid layer with two free surfaces when the adjoining medium is assumed to be electrically nonconducting, the case having some relevance to the ionospheric layer. The appropriate boundary conditions for this case are:

$$w = D^2\,w = 0\,, \;\; \Theta = 0\,, \;\; D\,\zeta = 0\,, \;\; \xi = 0 \;\text{ at } z = 0, 1 \text{ and}$$
$$\bar{h} \text{ is continuous.}$$

Making use of the above boundary conditions in $(17) - (21)$, we obtain

$$D^4\,w = 0\,, \;\; D^2\,\Theta = 0\,, \;\; \zeta = D^2\,\zeta = 0 \;\; \text{at} \;\; z = 0 \text{ and } 1.$$

Differentiating (17) twice with respect to z and using the above boundary conditions through Eqs. $(18) - (21)$, we get

$$D^6\,w = 0\,, \;\; D^4\,\Theta = 0\,, \;\; D^4\,\zeta = 0 \;\; \text{at} \;\; z = 0 \text{ and } 1.$$

This process can be continued and it can be shown that all the even derivatives of w must vanish for $z = 0, 1$. The proper solution of Eq. (22) characterizing the lowest mode is therefore

$$w = A\sin\pi z \qquad (23)$$

where A is a constant. Substituting the solution (23) in Eq. (22) and letting

$$a^2 = \pi^2\,x\,, \;\; R_1 = R/\pi^4\,, \;\; Q_1 = Q/\pi^2\,, \qquad \text{we obtain}$$

$$R_1 = \frac{(1+x)^3}{x} + Q_1(1+x)\cos^2\Phi \qquad (24)$$
$$+ \frac{M\cos^2\Phi\,(1+x)^2\,(1+x-3\,x\cos^2\Phi)^2}{(1+x)^2 + Q_1\,x\cos^2\Phi}\,,$$

where $M = \nu_0{}^2/\nu^2$ is a nondimensional parameter.

Equation (24) expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters Q_1 and M.

In Fig. 1, we have plotted R_1 as a function of x for $Q_1 = 100$ and for three values of $M\,(= 100, 200, 500)$ for the set $\Phi = 0°$. Similarly R_1 against x for the same values of Q_1 and M, but for the sets $\Phi = 45°$ and $\Phi = 90°$ have been plotted in Figs. 2 and 3 respectively.
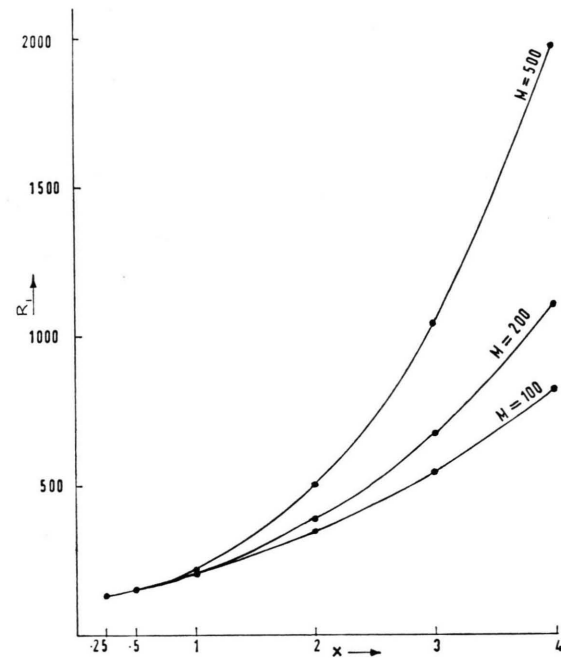

Fig. 1. The variation of R_1 with x for a fixed $Q_1 = 100$ and for three values of $M\,(= 100, 200, 500)$ for the set $\Phi = 0°$.

It is clear from Figs. 1 and 2 that, for a given value of x, R_1 increases continuously with increase in the value of M, thus showing the stabilizing effect of the finite Larmor radius. For the case $\Phi = 90°$ in Fig. 3, R_1 is independent of the effects of Q_1 and M, and a minimum exists at $x = \frac{1}{2}$.
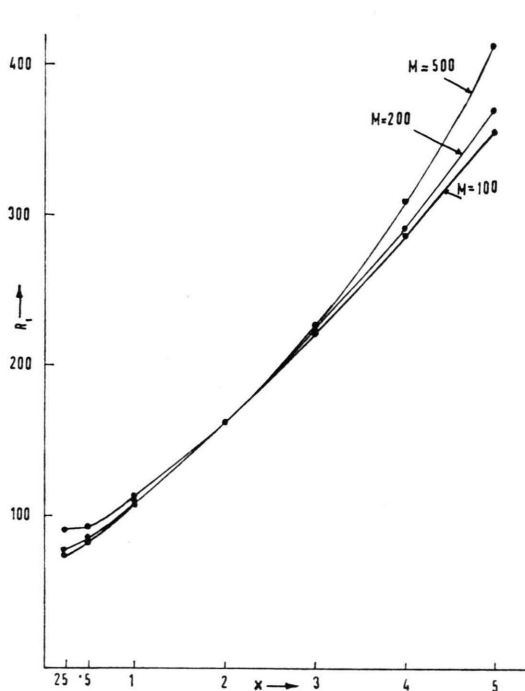
Fig. 2. The variation of R_1 with x for a fixed $Q_1 = 100$ and for three values of $M (= 100, 200, 500)$ for the set $\Phi = 45°$.
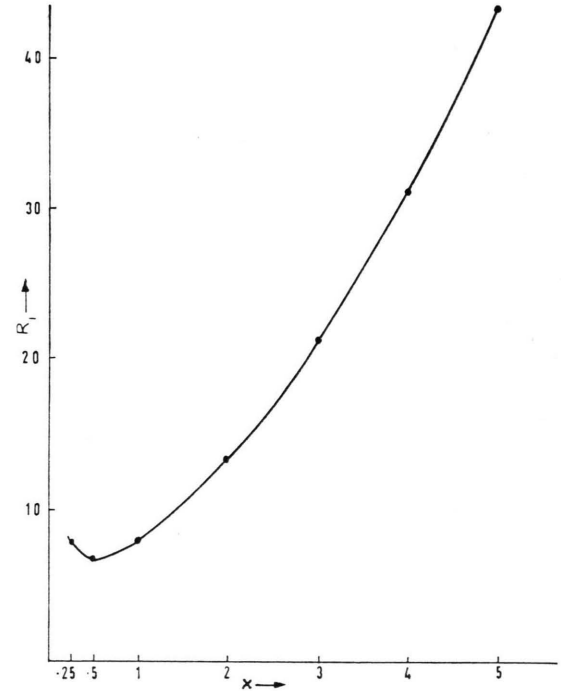
Fig. 3. The variation of R_1 with x for a fixed $Q_1 = 100$ for the set $\Phi = 90°$.

4. The Case of Overstability

In this section we discuss the possibility as to whether instability may occur as overstability. Here also we consider the case of two free boundaries.

Eliminating Θ, h_z, ξ and ζ from Eqs. $(12) - (16)$, we obtain

$$[\,(D^2 - a^2 - \sigma)\,(D^2 - a^2 - p_2\,\sigma)\,+ Q\,a^2\cos^2\Phi]\,[D^2 - a^2 - \sigma)\,(D^2 - a^2 - p_1\,\sigma)\,(D^2 - a^2 - p_2\,\sigma)$$
$$+ R\,a^2\,(D^2 - a^2 - p_2\,\sigma)\,+ Q\,a^2\cos^2\Phi\,(D^2 - a^2)\,(D^2 - a^2 - p_1\,\sigma)\,]\,w \qquad (25)$$
$$= \left(\frac{a^2\,\nu_0{}^2\cos^2\Phi}{\nu^2}\right)\,(D^2 - a^2 - p_1\,\sigma)\,(D^2 - a^2 - p_2\,\sigma)^2\,(D^2 - a^2 + 3\,a^2\cos^2\Phi)^2\,w\,,$$

where σ may be complex.

The appropriate solution of (25) characterizing the lowest mode is

$$w = A\sin\pi z\,. \qquad (26)$$

Substituting the solution (26) in (25) and letting $a^2 = \pi^2\,x$, $R_1 = R/\pi^4$, $Q_1 = Q/\pi^2$ and $i\,\sigma_1 = \sigma/\pi^2$, we get

$$R_1 = \left(\frac{1+x}{x}\right)\,(1 + x + i\,\sigma_1)\,(1 + x + i\,\sigma_1\,p_1)\,+ Q_1\cos^2\Phi\,\frac{(1+x)\,(1 + x + i\,\sigma_1\,p_1)}{1 + x + i\,\sigma_1\,p_2}$$
$$+ \frac{M\cos^2\Phi\,(1 + x + i\,\sigma_1\,p_1)\,(1 + x + i\,\sigma_1\,p_2)\,(1 + x - 3\,x\cos^2\Phi)^2}{(1 + x + i\,\sigma_1)\,(1 + x + i\,\sigma_1\,p_2)\,+ Q_1\,x\cos^2\Phi}\,. \qquad (27)$$

Since for overstability, our interest is to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it will suffice to find conditions for which (27) will admit of solutions with σ_1 real. Assuming, then, that σ_1 is real, and equating, separately, the real and the imaginary

parts of Eq. (27), we obtain

$$R_1 = \left(\frac{1+x}{x}\right)\left\{(1+x)^2 - p_1\sigma_1{}^2\right\} + \frac{Q_1\cos^2\Phi(1+x)\left\{(1+x)^2 + p_1 p_2 \sigma_1{}^2\right\}}{(1+x)^2 + p_2{}^2 \sigma_1{}^2}$$
$$+ \frac{L[\{(1+x)^2 - p_1 p_2 \sigma_1{}^2\}\{(1+x)^2 - p_2 \sigma_1{}^2 + Q_1 x \cos^2\Phi\} + \sigma_1{}^2(1+x)^2(1+p_2)(p_1+p_2)]}{\{(1+x)^2 - p_2\sigma_1{}^2 + Q_1 x \cos^2\Phi\}^2 + \sigma_1{}^2(1+x)^2(1+p_2)^2}, \tag{28}$$

and

$$0 = \left(\frac{1+x}{x}\right)(1+x)(1+p_1) + \frac{Q_1\cos^2\Phi(1+x)^2(p_1-p_2)}{(1+x)^2 + p_2{}^2\sigma_1{}^2}$$
$$+ \frac{L(1+x)\left[(1+x)^2(p_1-1) + p_2{}^2(p_1-1)\sigma_1{}^2 + (p_1+p_2)Q_1 x \cos^2\Phi\right]}{\{(1+x)^2 - p_2\sigma_1{}^2 + Q_1 x \cos^2\Phi\}^2 + \sigma_1{}^2(1+x)^2(1+p_2)^2}, \tag{29}$$

where

$$L = M\cos^2\Phi(1+x-3x\cos^2\Phi)^2.$$

Equation (29) can be written in the form

$$A_1\sigma_1{}^6 + B_1\sigma_1{}^4 + C_1\sigma_1{}^2 + D_1 = 0, \tag{30}$$

where

$$A_1 = \left(\frac{1+x}{x}\right)(1+x)(1+p_1)p_2{}^4,$$

$$B_1 = \left(\frac{1+x}{x}\right)(1+x)(1+p_1)\left\{(1+x)^2(2+p_2{}^2) - 2p_2 Q_1 x \cos^2\Phi\right\}p_2{}^2$$
$$+ Q_1\cos^2\Phi(1+x)^2 p_2{}^2(p_1-p_2) + L(1+x)p_2{}^4(p_1-1),$$

$$C_1 = \left(\frac{1+x}{x}\right)(1+x)(1+p_1)\left\{(1+x)^4(1+2p_2{}^2) + 2p_2(1+x)^2 Q_1 x \cos^2\Phi(p_2-1)\right.$$
$$+ p_2{}^2 Q_1{}^2 x^2\cos^4\Phi\} + Q_1\cos^2\Phi(1+x)^2(p_1-p_2)\left\{(1+x)^2(1+p_2{}^2)\right.$$
$$- 2p_2 Q_1 x \cos^2\Phi\} + p_2{}^2(p_1-1)(1+x)^2 + p_2{}^2\{(1+x)^2(p_1-1) + (p_1+p_2)Q_1 x\cos^2\Phi\}, \tag{31}$$

$$D_1 = \left(\frac{1+x}{x}\right)(1+x)^3(1+p_1)\left\{(1+x)^2 + Q_1 x\cos^2\Phi\right\}^2$$
$$+ Q_1\cos^2\Phi(1+x)^2(p_1-p_2)\left\{(1+x)^2 + Q_1 x\cos^2\Phi\right\}^2$$
$$+ L(1+x)^3\left\{(1+x)^2(p_1-1) + (p_1+p_2)Q_1 x\cos^2\Phi\right\}.$$

Equation (30) is cubic in $\sigma_1{}^2$. Applying Hurwitz' criterion to Eq. (30) we find, as all the coefficients of this equation are positive when $p_1 > p_2 > 1$ and $Q_1{}^2\cos^4\Phi < 16$, that either all the three roots of this equation are real and negative or there is one negative real root and the remaining two roots are complex with negative real parts.

Hence (30) will have all values of $\sigma_1{}^2$ such that $Re(\sigma_1{}^2) < 0$ if

$$p_1 > p_2 > 1, \quad Q_1{}^2\cos^4\Phi < 16, \tag{32}$$

which means

$$\nu > \eta > \varkappa \quad \text{and} \quad \frac{\mu_e H^2 d^2}{4\pi\varrho\nu\eta} < 4\pi^2\sec^2\Phi. \tag{33}$$

For overstability σ_1 has to be real and it has been shown that not a single value of $\sigma_1{}^2$ is positive. Equation (33) are, therefore, the sufficient conditions for the nonexistence of overstability, the violation of which does not necessarily imply occurrence of overstability.

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